

# LOCATING THE ZEROS OF PARTIAL SUMS OF $e^z$ WITH RIEMANN-HILBERT METHODS

T. KRIECHERBAUER, A. B. J. KUIJLAARS, K. D. T-R MCLAUGHLIN, AND P. D. MILLER

*Dedicated to Percy Deift with gratitude and admiration.*

**ABSTRACT.** In this paper we derive uniform asymptotic expansions for the partial sums of the exponential series. We indicate how this information will be used in a later publication to obtain full and explicitly computable asymptotic expansions with error bounds for all zeros of the Taylor polynomials  $p_{n-1}(z) = \sum_{k=0}^{n-1} z^k / k!$ . Our proof is based on a representation of  $p_{n-1}(nz)$  in terms of an integral of the form  $\int_{\gamma} \frac{e^{n\phi(s)}}{s-z} ds$ . We demonstrate how to derive uniform expansions for such integrals using a Riemann-Hilbert approach. A comparison with classical steepest descent analysis shows the advantages of the Riemann-Hilbert analysis in particular for points  $z$  that are close to the critical points of  $\phi$ .

## 1. INTRODUCTION

During the past fifteen years and largely due to the ground breaking work [5], [6] of Deift and Zhou, Riemann-Hilbert problems have become a powerful tool in asymptotic analysis with applications in many fields such as inverse scattering theory, integrable PDE's, orthogonal polynomials, statistical mechanics and random matrix theory. With this paper we would like to add another item to this list of applications, namely the study of the asymptotic behavior of zeros of Taylor polynomials of entire functions. Although our method works in principle for a large class of entire functions we will restrict our attention to the classic case of Taylor polynomials of  $e^z$  in order to keep the presentation as non technical as possible. It is also one of the intentions of this paper to highlight the advantageous aspects of the Riemann-Hilbert approach in a simple situation. The first and maybe most surprising feature is how many objects of interest can be characterized as the unique solution of some Riemann-Hilbert problem and with this paper we present one more

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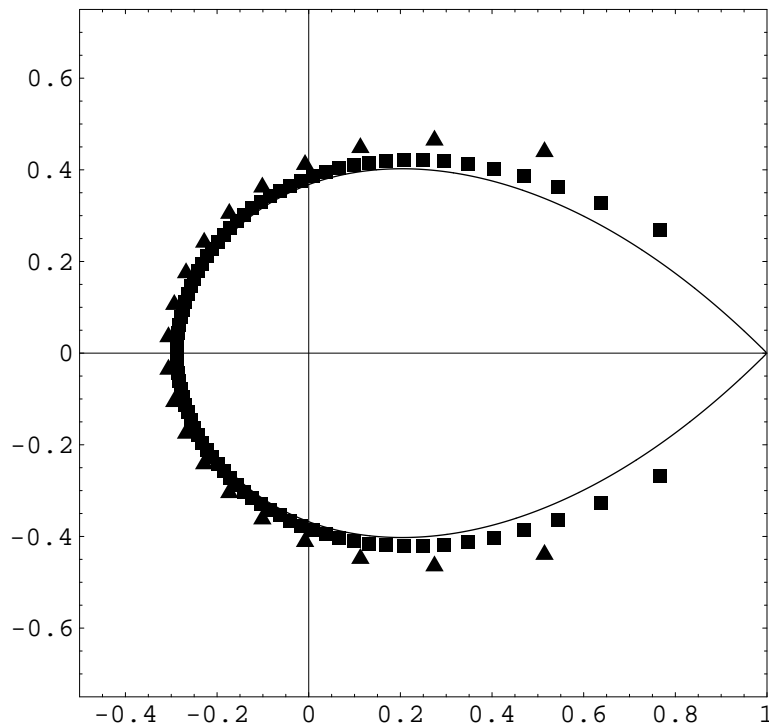


FIGURE 1. Szegő curve (solid line) and rescaled zeros of  $p_{20}$  (triangles) and  $p_{80}$  (boxes)

illustration of this fact. Secondly, there is a set of techniques to transform Riemann-Hilbert problems to some form where an asymptotic expansion of the quantity of interest can be extracted in a systematic fashion. In our case, we will only need one of these techniques and that is the construction of a local parametrix. This construction exemplifies in a very simple situation just one of the wonderful ideas of Deift and Zhou [5], [6], namely how to effectively localize a Riemann-Hilbert problem which by definition is a global problem in the complex plane. Localization is an indispensable step in the asymptotic analysis of any Riemann-Hilbert problem where the asymptotic behavior of the solution is different for different regions of the complex plane. Moreover, and that is the third aspect we want to highlight, the localization procedure of Deift and Zhou always provides for matching asymptotics in those regions where the asymptotic behavior changes and one obtains error bounds uniform in all of  $\mathbb{C}$ .

We denote by  $p_n(z) := 1 + z + \cdots + \frac{z^n}{n!}$  the partial sums of the exponential series. The problem to describe the asymptotic distribution of the zeros of  $p_n$  was posed and solved in the classical paper of Szegő [11]. He proved that the zeros of  $p_n$ , divided by  $n$ , converge in the limit  $n \rightarrow \infty$  to some curve  $D_\infty$ , now called Szegő curve, which consists of all complex numbers  $|z| \leq 1$  that satisfy the equation  $|ze^{1-z}| = 1$ . The curve  $D_\infty$  together with the rescaled zeros of  $p_n$  for  $n = 20$  and  $n = 80$  are displayed in Figure 1. Moreover, Szegő also determined the

limiting distribution of the rescaled zeros on  $D_\infty$ . A first result presenting error bounds on the distance between the Szegő curve and the zeros of  $p_n(nz)$  has been established by Buckholtz [3] who showed that they are located in the exterior of  $D_\infty$  at a distance of at most  $2e/\sqrt{n}$ . For each zero of  $p_n(nz)$  its distance from  $D_\infty$  is measured by the minimal distance between the zero and all the points of the Szegő curve. Subsequently more detailed asymptotics of  $p_n(nz)$  have been derived. It turns out that  $z_0 = 1$  is a critical point where the asymptotic behavior changes. It was shown by Newman and Rivlin [9] that in neighborhoods of  $z_0$  of size  $O(1/\sqrt{n})$  the rescaled polynomials  $p_n(n + w\sqrt{n})$  can asymptotically be expressed in terms of the complementary error function  $\operatorname{erfc}$ . Carpenter, Varga, and Waldvogel [4] (see also [13] and see [12] for an interesting discussion of zeros of the partial sums of  $e^z$ ) then provided an asymptotic expansion of  $p_n(nz)$  in compact subsets of  $\mathbb{C} \setminus \{z_0\}$ . These results were used in [4] to obtain lower and upper bounds on the distance of the zeros from the Szegő curve. Note that up to the recent paper by Bleher and Mallison [2] no uniform asymptotics for  $p_n(nz)$  and its zeros were available in a fixed size neighborhood of the critical point  $z_0$ . In a different direction and using methods from logarithmic potential theory Andrievskii, Carpenter and Varga [1] recently extended the results of Szegő [11] on the angular distribution of the zeros by proving uniform error bounds for regions including  $z_0$ . We refer the reader to the recent review [14] for a description of more results on the zeros of  $p_n$ . Related results on the zeros of the partial sums of  $\cos$ ,  $\sin$  and of more general sums of exponential functions can be found in [2] and in references therein. Finally, we mention the review [10] on the behavior of the zeros of more general sections and tails of power series.

The main novel result proved in the present paper is Theorem 3.3 which provides an explicitly computable asymptotic expansion near  $z_0$  for a quantity  $F_n(z)$  closely related to  $p_{n-1}(nz)$  (see (1.1), (1.2) below). Together with Remark 3.4 we produce an asymptotic expansion of  $F_n$  with uniform error bounds for  $z$  contained in some region  $V$  where all the roots of  $p_{n-1}(nz)$  are located. These results will be used in a subsequent publication to derive explicitly computable asymptotic expansions for all the zeros of  $p_{n-1}$  in terms of the zeros of the complementary error function for zeros close to the critical point, otherwise in terms of the solutions of  $(ze^{1-z})^n = 1$  which lie on the Szegő curve. We will state the corresponding results without proof in Theorems 4.1 and 4.2 below.

Let  $\gamma$  be any smooth Jordan curve encircling the origin counterclockwise. For  $z \in \mathbb{C} \setminus \gamma$  define

$$(1.1) \quad F_n(z) := \frac{1}{2\pi i} \int_{\gamma} \frac{s^{-n} e^{n(s-1)}}{s-z} ds = \frac{1}{2\pi i} \int_{\gamma} \frac{e^{n\phi(s)}}{s-z} ds, \quad \text{with } \phi(s) = s - 1 - \ln s,$$

where  $\ln$  denotes the standard branch of the logarithm slit along the negative real axis. A straightforward application of the calculus of residues then yields

$$(1.2) \quad (ez)^{-n} p_{n-1}(nz) = \begin{cases} -F_n(z), & \text{for } z \text{ in the exterior of } \gamma \\ e^{n\phi(z)} - F_n(z), & \text{for } z \neq 0 \text{ in the interior of } \gamma \end{cases}$$

To the best of our knowledge the Cauchy-type representation of  $p_{n-1}$  provided by (1.1) and (1.2) which is crucial for our Riemann-Hilbert approach has so far not been used in the asymptotic analysis of the zeros of  $p_{n-1}$ .

Given the form of  $F_n$  it is natural to use the method of steepest descent for the asymptotic analysis of the integral. This method requires that the path  $\gamma$  of

integration passes through the critical point  $z_0 = 1$  of the function  $\phi$ . This indicates already that the asymptotic analysis will be most difficult for  $z$  close to 1 due to the term  $(s - z)^{-1}$  in the integrand. We discuss this approach in detail in Section 2 below and we will see that by standard techniques of steepest descent analysis one may derive an asymptotic expansion for  $F_n(z)$  with uniform error bounds for  $|z - 1| > n^{-\alpha}$  and  $0 < \alpha < 1/2$ .

Section 3 contains the central result of this paper where we derive an asymptotic expansion for  $F_n$  in some fixed neighborhood  $U_\epsilon(1)$  of the critical point. Here we use that  $F_n$  can easily be characterized as the unique solution of a Riemann-Hilbert problem since by definition  $F_n$  is just the Cauchy transform of  $e^{n\phi}$  (see  $(RHP)_1$  described at the beginning of Section 3). Using the standard Riemann-Hilbert technique of constructing a local parametrix near the critical point the derivation of the asymptotic expansion is no more difficult than the computations in the non critical situation of Section 2.

In the final Section 4 we briefly explain how to use the asymptotic results to obtain asymptotic information on the location of the zeros. This, however, is a somewhat technical affair and we will present a complete account of all results and their proofs in a later publication. In Theorems 4.1, 4.2 we state our results for the zeros of  $p_{n-1}$  in the upper half plane.

## 2. CLASSICAL STEEPEST DESCENT ANALYSIS

In order to remind ourselves of the method of steepest descent (see also [8] for an elementary exposition) we first determine the large  $n$  asymptotics of a quantity that is related to  $F_n$  defined in (1.1) but is somewhat simpler to analyze. Let

$$(2.1) \quad G_n := \frac{1}{2\pi i} \int_{\gamma} e^{n\phi(s)} ds = \frac{1}{2\pi i} \int_{\gamma} s^{-n} e^{n(s-1)} ds,$$

where  $\gamma$  denotes some smooth Jordan curve that is oriented counterclockwise containing the origin in its interior.

The method of steepest descent offers a recipe how to deform the contour of integration in such a way that the asymptotic behavior of  $G_n$  can be determined most conveniently. More precisely, we are to consider contours that pass through critical points of  $\phi$  along the path of steepest descent with respect to the real part of  $\phi$ . Such a choice of the contour ensures that the modulus of the integrand obtains its local maximums only at the critical points of  $\phi$  and – up to exponentially small error terms – the integral is determined by the contributions of those parts of the contour which lie in small neighborhoods of the critical points. In the situation at hand the function  $\phi$  has only one critical point at  $z_0 = 1$  and we can easily understand the behavior of the real part of  $\phi$ . It has a saddle point at  $z_0 = 1$  and the solid curve in Figure 2 displays those numbers  $z$  satisfying  $\Re(\phi(z)) = \Re(\phi(z_0)) = 0$ . Note, that the Szegő curve  $D_\infty$  which was displayed in Figure 1 above coincides with the closed loop part of the solid curve. Moreover, the dotted line in this figure shows the path of steepest descent away from the saddle. This path can easily be determined to consist of those points in  $\mathbb{C} \setminus \{0\}$  satisfying  $\arg z = \Im z$  by using the property that the imaginary part of  $\phi$  remains constant on it, i.e.  $\Im(\phi(z)) = \Im(\phi(z_0)) = 0$ . Choosing the smooth closed contour  $\tilde{\gamma}$  to coincide with the path of steepest descent in the right half plane and with the dashed line in the left half plane (in the left

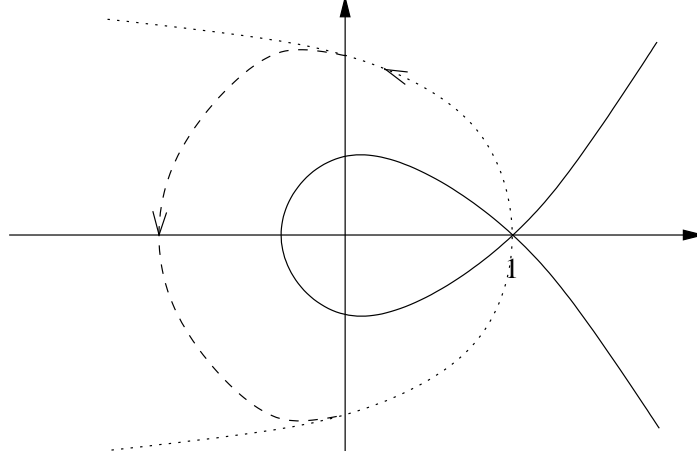


FIGURE 2.

half plane any smooth curve that does not intersect the solid line will do) we obtain

$$(2.2) \quad G_n = \frac{1}{2\pi i} \int_{\tilde{\gamma}} e^{n\phi(s)} ds = \frac{1}{2\pi i} \int_{\tilde{\gamma} \cap U} e^{n\phi(s)} ds + O(e^{-nc}),$$

where  $U$  is any fixed neighborhood of  $z_0 = 1$ . It is only the number  $c > 0$  in the error term that will depend on the choice of  $U$ .

In order to further analyze the integral along  $\tilde{\gamma} \cap U$  we change variables near the critical point of  $\phi$ . Since  $\phi''(z_0) \neq 0$  we can find an open neighborhood  $U_0$  of  $z_0$  and a biholomorphic map  $\lambda : U_{\delta_0}(0) \rightarrow U_0$  for some  $\delta_0 > 0$  such that

$$(2.3) \quad \phi(\lambda(\xi)) = \xi^2 \quad \text{for } \xi \in U_{\delta_0}(0)$$

Observe that  $\lambda$  maps the imaginary axis onto the path of steepest descent, more precisely  $\lambda(i\mathbb{R} \cap U_{\delta_0}(0)) = \tilde{\gamma} \cap U_0$ . This can be seen from the characterization of the path of steepest descent by the imaginary part of  $\phi$ , i.e.

$$\tilde{\gamma} \cap U_0 = \{z \in U_0 : \Im \phi(z) = \Im \phi(1) = 0 \text{ and } \Re \phi(z) \leq \Re \phi(1) = 0\}.$$

Choosing  $U := U_0$  in (2.2) we obtain

$$G_n = \frac{1}{2\pi i} \int_{-i\delta_0}^{i\delta_0} e^{n\xi^2} \lambda'(\xi) d\xi + O(e^{-nc}) = \frac{1}{2\pi\sqrt{n}} \int_{-\delta_0\sqrt{n}}^{\delta_0\sqrt{n}} e^{-t^2} \lambda'(it/\sqrt{n}) dt + O(e^{-nc})$$

Expanding  $\lambda$  at  $\xi = 0$  leads to an asymptotic expansion for  $G_n$ . For example, we learn from

$$\lambda(\xi) = 1 + \sqrt{2}\xi + \frac{2}{3}\xi^2 + \frac{\sqrt{2}}{18}\xi^3 + O(\xi^4)$$

that

$$G_n = \frac{1}{\sqrt{2\pi n}} \left( 1 - \frac{1}{12}n^{-1} + O(n^{-2}) \right).$$

Observe that the calculus of residues implies  $G_n = e^{-n}n^{n-1}/\Gamma(n)$  directly from (2.1) and we have thus found some version of Stirling's formula.

It is clear that we may extend our reasoning to integrals of the form

$$(2.4) \quad \frac{1}{2\pi i} \int_{\gamma} h(s) e^{n\phi(s)} ds$$

for analytic functions  $h$ . Up to exponentially small error terms (2.4) is given by

$$\frac{1}{2\pi\sqrt{n}} \int_{-\delta_0\sqrt{n}}^{\delta_0\sqrt{n}} h(\lambda(it/\sqrt{n})) e^{-t^2} \lambda'(it/\sqrt{n}) dt.$$

This expression can easily be expanded in powers of  $1/n$  using the expansions of  $\lambda$  around 0 and of  $h$  around 1. For example, in the case  $h(1) \neq 0$  we immediately obtain

$$(2.5) \quad \frac{1}{2\pi i} \int_{\gamma} h(s) e^{n\phi(s)} ds = \frac{h(1)}{\sqrt{2\pi n}} (1 + O(n^{-1}))$$

for any Jordan curve winding counterclockwise around the origin.

We are now in a position to apply our reasoning to  $F_n$  by replacing the analytic function  $h$  in (2.5) by the meromorphic functions  $h_z(s) = (s - z)^{-1}$ . Indeed, we have

$$F_n(z) = \frac{1}{2\pi i} \int_{\gamma} h_z(s) e^{n\phi(s)} ds.$$

Before formulating a result on the large  $n$  behavior of  $F_n(z)$  we specify the type of contours  $\gamma$  to be considered.

**Definition 2.1.** A contour  $\gamma$  is said to be *admissible* if

- (i)  $\gamma$  is a smooth Jordan curve winding counterclockwise around the origin, and
- (ii)  $\gamma$  has a positive distance from the solid curve in Figure 2, except for a part that lies in some neighborhood  $U$  of  $z_0 = 1$ . In this set  $U$  the contour  $\gamma$  coincides with the path of steepest descent (dotted line in Figure 2).

The steepest descent analysis described above then gives

**Theorem 2.2.** *For any admissible contour  $\gamma$  and  $z \in \mathbb{C} \setminus \gamma$  we have*

$$F_n(z) = \frac{1}{\sqrt{2\pi n}(1 - z)} (1 + O(n^{-1})) ,$$

where the error term is uniform for  $z \in \mathbb{C} \setminus (U_{\epsilon}(1) \cup \gamma)$  for any  $\epsilon > 0$ .

*Proof.* It remains to discuss the uniformity of the error bound. The only step in the derivation of the asymptotic formula for which the uniformity appears problematic is the estimate used in (2.2) for points  $z$  that lie arbitrarily close to  $\gamma$  because of the  $(s - z)^{-1}$  term in the integrand. However, since  $z$  stays away some fixed distance  $\epsilon$  from the critical point  $z_0$ , one may deform the contour of integration for each  $z$  in such a way that the value of  $F_n$  does not change and such that the deformed contour is separated from both, the solid line in Figure 2 and from  $z$  by a minimal distance that only depends on  $\epsilon$ .  $\square$

A somewhat more refined analysis provides uniform error bounds also outside shrinking disks  $|z - 1| > n^{-\alpha}$  for any fixed  $0 < \alpha < 1/2$ . In this case the error term  $O(n^{-1})$  in Theorem 2.2 has to be replaced by  $O(n^{-1}|1 - z|^{-2})$ . For values of  $z$  that lie in shrinking discs  $|z - 1| = O(n^{-\alpha})$  it is still possible but cumbersome to use the method described above for computing the asymptotics of  $F_n(z)$ . It is precisely this

situation in which we want to demonstrate how ideas from the asymptotic analysis of Riemann-Hilbert problems can be used.

### 3. RIEMANN-HILBERT ANALYSIS

In this section we utilize that the representation (1.1), (1.2) of  $p_{n-1}$  is of Cauchy-type which enables us to employ Riemann-Hilbert techniques. The key for this is that the function  $F_n$  defined in (1.1) is the Cauchy transform of  $e^{n\phi}$  with respect to the contour  $\gamma$ . Hence, and this is the feature of the Cauchy transform that provides the link to Riemann-Hilbert problems,  $F_n$  is analytic in  $\mathbb{C} \setminus \gamma$  and the values of  $F_n(z)$  differ by  $e^{n\phi(s)}$  as  $z$  approaches  $s \in \gamma$  from opposite sides. Moreover, these properties of  $F_n$  together with the behavior of  $F_n(z)$  as  $z \rightarrow \infty$  characterize  $F_n$  uniquely. More precisely, we will show in Lemma 3.1 below that  $F_n$  is the unique solution of the following scalar Riemann-Hilbert problem  $(RHP)_1$ :

*Given an admissible contour  $\gamma$  and  $n \in \mathbb{N}$ . Seek an analytic function  $Y: \mathbb{C} \setminus \gamma \rightarrow \mathbb{C}$  such that*

- (i)  $Y_+(s) = Y_-(s) + e^{n\phi(s)}$  for  $s \in \gamma$  ,
- (ii)  $Y(z) \rightarrow 0$  for  $|z| \rightarrow \infty$  .

Condition (i) is shorthand notation for the requirement that  $Y$  has continuous extensions from the interior of  $\gamma$  (respectively from the exterior of  $\gamma$ ) onto  $\gamma$  which are denoted by  $Y_+$  (respectively  $Y_-$ ) and which satisfy relation (i). As mentioned above the question of existence and uniqueness of solutions for this Riemann-Hilbert problem is answered by the following

**Lemma 3.1.**  *$F_n$  as defined in (1.1) is the unique solution of  $(RHP)_1$ .*

*Proof.* Using (1.2) it is easy to verify that  $F_n$  indeed solves  $(RHP)_1$ . In order to prove uniqueness one first shows that the difference  $\Delta$  of two solutions of  $(RHP)_1$  is continuous across  $\gamma$  and hence entire. Liouville's theorem together with condition (ii) then implies  $\Delta(z) = 0$  for all  $z \in \mathbb{C}$ .  $\square$

Next we apply the method of constructing a local parametrix for the Riemann-Hilbert problem in order to resolve the difficulties at the critical point  $z_0$ . We recall from Section 2 that the change of variables  $z \rightarrow t$  with  $z = \lambda(it/\sqrt{n})$ ,  $z \in U_0$ , maps the contour of steepest descent into the real axis and transforms  $n\phi$  to normal form,  $n\phi(z) = -t^2$ . This motivates the definitions

$$(3.1) \quad h(\zeta) := \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{e^{-u^2}}{u - \zeta} du, \quad \zeta \in \mathbb{C} \setminus \mathbb{R},$$

$$(3.2) \quad P_n(z) := h(-i\sqrt{n}\lambda^{-1}(z)), \quad z \in U_0 \setminus \gamma.$$

We have  $h_+(t) = h_-(t) + e^{-t^2}$  for  $t \in \mathbb{R}$ , where  $h_{\pm}(t)$  denotes  $\lim_{\eta \rightarrow 0+} h(t \pm i\eta)$ . This relation is a consequence of standard properties of the Cauchy transform. Alternatively, since  $h$  is the Cauchy transform of an analytic function  $e^{-u^2}$ , one may also derive this relation using only the calculus of residues. Substituting  $s = \lambda(it/\sqrt{n})$  we obtain

$$(P_n)_+(s) = (P_n)_-(s) + e^{n\phi(s)} \quad \text{for all } s \in U_0 \cap \gamma.$$

$P_n$  is thus a local solution of the Riemann-Hilbert problem  $(RHP)_1$  in  $U_0$ . Moreover,  $P_n$  is of a rather explicit nature, since  $h$  is related to the well studied complementary error function [7] (see also (4.5) below) and the Taylor coefficients of  $\lambda$  can

be computed explicitly at  $\xi = 0$  to all orders from the defining relation (2.3). Note, however, that this local solution  $P_n$  cannot be continued to the global solution because  $\lambda^{-1}(z)$  has singularities outside of  $U_0$ . The procedure to take advantage of this local parametrix is strikingly simple. Choose  $\epsilon > 0$  such that the closed disc  $\overline{U_{2\epsilon}(1)}$  is contained in  $U_0$ . We then set

$$(3.3) \quad \tilde{m}(z) := \begin{cases} Y(z), & \text{for } z \in \mathbb{C} \setminus (\gamma \cup \overline{U_{2\epsilon}(1)}) \\ Y(z) - P_n(z), & \text{for } z \in U_{2\epsilon}(1) \setminus \gamma \end{cases}$$

where  $Y = F_n$  denotes the unique solution of  $(RHP)_1$ . Observe that  $\tilde{m}_+(s) = \tilde{m}_-(s)$  for  $s \in \gamma \cap U_{2\epsilon}(1)$  since the jumps of  $Y$  and  $P_n$  cancel each other. We may therefore extend  $\tilde{m}$  within  $U_{2\epsilon}(1)$  to an analytic function. We denote this function with a slightly extended domain of definition by  $m$ . It is obvious that  $m$  again solves a Riemann-Hilbert problem. More precisely, denote

$$\Gamma_1 := \partial U_{2\epsilon}(1), \quad \Gamma_2 := \gamma \setminus U_{2\epsilon}(1), \quad \Gamma := \Gamma_1 \cup \Gamma_2.$$

The dashed line in Figure 3 provides a sketch of  $\Gamma$ . It is straightforward to verify that the function  $m$  is a solution of  $(RHP)_2$ :

*Seek an analytic function  $M : \mathbb{C} \setminus \Gamma \rightarrow \mathbb{C}$  such that*

- (i)  $M_+(s) = M_-(s) - P_n(s)$  for  $s \in \Gamma_1 \setminus \Gamma_2$   
 $M_+(s) = M_-(s) + e^{n\phi(s)}$  for  $s \in \Gamma_2$
- (ii)  $M(z) \rightarrow 0$  for  $|z| \rightarrow \infty$ .

As above  $M_{\pm}$  denote the continuous extensions of  $M$  from the interior (+) and exterior (−). We show in Remark 3.2 below that again we can use the Cauchy transform to solve the Riemann-Hilbert problem  $(RHP)_2$ :

$$(3.4) \quad m(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{m_+(s) - m_-(s)}{s - z} ds = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{-P_n(s)}{s - z} ds + \frac{1}{2\pi i} \int_{\Gamma_2} \frac{e^{n\phi(s)}}{s - z} ds.$$

The orientations of  $\Gamma_1$  and  $\Gamma_2$  are chosen to be counterclockwise so that the + side always lies to the left of the contour. Observe that all  $z \in U_{\epsilon}(1)$  have at least distance  $\epsilon$  from  $\Gamma$  so that the contour of integration has been moved away from the singularity  $(s - z)^{-1}$  of the integrand. Moreover, the integral over  $\Gamma_2$  is exponentially small so that the asymptotic expansion of  $m$  is solely determined by the integral over  $\Gamma_1$ . Before we formulate our theorem on the asymptotics of  $F_n(z)$  for  $z$  near  $z_0 = 1$  we present an elementary proof of (3.4) which does not make use of the fact that  $m$  is the unique solution of  $(RHP)_2$ . Nevertheless,  $(RHP)_2$  provides a clear explanation why (3.4) holds: The solution of this scalar Riemann-Hilbert problem is simply given by the Cauchy transform of the jump  $m_+ - m_-$  on  $\Gamma$ .

*Remark 3.2.* One may show the first equality of (3.4) by applying the calculus of residues to

$$\int_{\sigma_1} \frac{m(s)}{s - z} ds + \int_{\sigma_2} \frac{m(s)}{s - z} ds + \int_{\sigma_3} \frac{m(s)}{s - z} ds$$

where  $\sigma_1$ ,  $\sigma_2$  and  $\sigma_3$  denote the closed curves shown in Figure 3 and by taking the limit as these curves approach  $\Gamma$  (dashed line). Note that condition (ii) in  $(RHP)_2$  implies that  $m(s)/(s - z)$  has a vanishing residue at  $s = \infty$ .

We are now ready to state our main result.



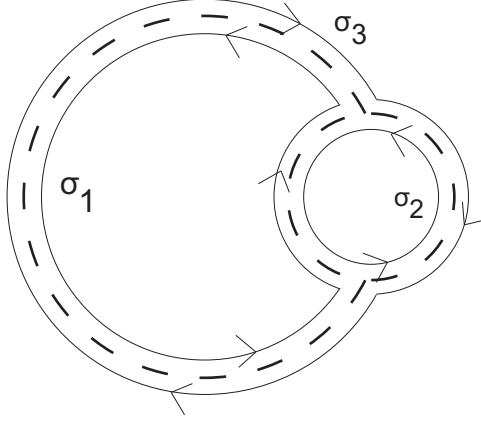


FIGURE 3.

**Theorem 3.3.** *There exists  $\epsilon > 0$  and functions  $g_j$  analytic in  $U_\epsilon(1)$  such that for any admissible curve  $\gamma$  and any  $r \in \mathbb{N}$  we have*

$$(3.5) \quad F_n(z) = P_n(z) + \frac{1}{\sqrt{2\pi n}} \left( \sum_{j=0}^{r-1} \frac{g_j(z)}{n^j} + O\left(\frac{1}{n^r}\right) \right),$$

where the error term is uniform for  $z \in U_\epsilon(1) \setminus \gamma$  and the Taylor coefficients of all  $g_j$  are explicitly computable at  $z_0 = 1$ , e.g.

$$g_0(z) = \frac{1}{3} - \frac{1}{12}(z-1) + O((z-1)^2).$$

*Proof.* Choose  $\epsilon > 0$  such that the closed disc  $\overline{U_{2\epsilon}(1)}$  is contained in  $U_0 \cap U$  where  $U_0$  is defined above (2.3) and  $U$  is determined by the curve  $\gamma$  (cf. Definition 2.1(ii)). From the discussion of the present section (see in particular (3.3), (3.4)) it follows that the unique solution  $F_n = Y$  of  $(RHP)_1$  can be written for  $z \in U_\epsilon(1)$  in the form

$$F_n(z) = P_n(z) + m(z) = P_n(z) - \frac{1}{2\pi i} \int_{\Gamma_1} \frac{P_n(s)}{s-z} ds + O(e^{-nc}).$$

Observe that we may not use the calculus of residues to evaluate  $\int_{\Gamma_1} \frac{P_n(s)}{s-z} ds$  since  $P_n$  is not a meromorphic function. Nevertheless, using

$$\frac{1}{u-\zeta} = - \sum_{j=0}^{2r} \frac{u^j}{\zeta^{j+1}} + \frac{u^{2r+1}}{\zeta^{2r+1}(u-\zeta)}$$

together with the fact that the Cauchy transform  $\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{s^{2r+1} e^{-s^2}}{s-z} ds$  is bounded on  $\mathbb{C} \setminus \mathbb{R}$  (which can e.g. be proved using a contour deformation argument) one derives that for each  $r \in \mathbb{N}$  there exists a constant  $C$  such that

$$\left| h(\zeta) + \frac{1}{2\pi i \zeta} \sum_{j=1}^r \frac{\Gamma(j + \frac{1}{2})}{\zeta^{2j}} \right| \leq \frac{C}{|\zeta|^{2r+1}}$$

for all  $\zeta \in \mathbb{C} \setminus \mathbb{R}$ . From the definition of  $P_n$  in terms of  $h$  we obtain

$$(3.6) \quad P_n(s) = \frac{1}{\sqrt{2\pi n}} \left( \sum_{j=0}^r \frac{G_j(s)}{n^j (s-1)^{2j+1}} + \text{rest} \right).$$

By choosing  $\epsilon$  smaller, if necessary, we can ensure that all  $G_j$  are analytic and non-zero in some open neighborhood of  $\overline{U_{2\epsilon}(1)}$ . The term “rest” in (3.6) is of order  $n^{-r}$  uniformly for  $s \in \partial U_{2\epsilon}(1)$ . The calculus of residues then leads to the desired representation of  $F_n$  and the Taylor coefficients of  $g_j$  can be derived from the Taylor coefficients of  $\lambda^{-1}$  by explicit computation.  $\square$

*Remark 3.4.* Replacing  $U_{2\epsilon}(1)$  by  $U_{\epsilon/2}(1)$  in (3.3) and adapting the arguments above one obtains an expansion of  $F_n(z)$  also for  $|z-1| > \epsilon$ . Indeed, one can prove that there exist polynomials  $h_j$  of degree  $2j$  such that for all  $r \in \mathbb{N}$  we have

$$(3.7) \quad F_n(z) = \frac{1}{\sqrt{2\pi n}(1-z)} \left( 1 + \sum_{j=1}^{r-1} \frac{h_j(z)}{n^j (z-1)^{2j}} + O\left(\frac{1}{n^r}\right) \right),$$

where the error term is uniform for  $\epsilon \leq |z-1| \leq 2$ . Again the polynomials  $h_j$  may be computed explicitly, e.g.  $h_1(z) = -\frac{1}{12}(z^2 + 10z + 1)$ . This result sharpens the statement of Theorem 2.2. It is also not difficult to see that the expansion (3.7) holds outside shrinking circles  $n^{-\alpha} \leq |z-1| \leq 2$  for any fixed  $0 < \alpha < 1/2$ . In this case the error term has to be replaced by  $O\left(\frac{1}{n^r |1-z|^{2r}}\right)$  to ensure uniformity in  $z$ .

#### 4. LOCATING THE ZEROS OF THE EXPONENTIAL SUM

The arguments and results put forward in this section will be presented in detail in a later publication, where we will construct for a specific admissible contour  $\gamma$  precisely  $n-1$  different solutions (together with their asymptotic expansions) of the equation

$$(4.1) \quad e^{n\phi(z)} = F_n(z)$$

in the interior of  $\gamma$ , provided  $n$  is sufficiently large. Since  $p_{n-1}(nz)$  has at most  $n-1$  different zeros (1.2) implies that we have located all the zeros of  $p_{n-1}$ . We will find the solutions of (4.1) by showing for each  $1 \leq k \leq n-1$  that the equation

$$(4.2) \quad G_n(z) := \tilde{\phi}(z) - \frac{1}{n} \ln F_n(z) = -\frac{2\pi i k}{n}$$

has one solution in the unit disc  $U_1(0)$ . Here  $\tilde{\phi}$  is defined as  $\phi$  in (1.1) with the only difference that the branch of the logarithm is now chosen in such a way that its imaginary part takes values in  $(0, 2\pi)$  rather than in  $(-\pi, \pi)$ . Existence and asymptotic expansions of the roots of (4.2) are obtained by a standard procedure. First one constructs solutions  $\alpha_{k,n}$  of

$$(4.3) \quad A_n(z) = -\frac{2\pi i k}{n}$$

for some approximation  $A_n$  of  $G_n$ . Then a contraction mapping argument will be used to conclude that the original equation (4.2) has a solution  $z_{k,n}$  close to  $\alpha_{k,n}$ . Since we can prove uniform bounds on the derivatives of the error terms in (3.5), (3.7) we obtain in addition an asymptotic expansion for  $z_{k,n}$  in terms of  $\alpha_{k,n}$ .

**4.1. Zeros away from the critical point.** In this case it suffices to use the crude approximation

$$A_n(z) = \tilde{\phi}(z) = z - 1 - \tilde{\ln}(z),$$

where  $\tilde{\ln}$  denotes the branch of the logarithm described above. The solutions  $\alpha_{k,n}$  of (4.3) all lie on the Szegő curve  $D_\infty$ . One can show that the distance of  $\alpha_{k,n}$  and  $z_{k,n}$  is of the same order as  $A_n - G_n = \frac{1}{n}(\ln F_n) = O((\ln n)/n)$  (see Theorem 2.2). Note that this result only holds for  $z_{k,n}$  that lie in a compact subset of  $\mathbb{C} \setminus \{1\}$ . An extended version of Theorem 2.2 as discussed at the end of Section 2 allows to include also those solutions  $z_{k,n}$  that lie outside shrinking discs  $|z - 1| \geq n^{-\alpha}$  with  $0 < \alpha < 1/2$ . The distance between  $\alpha_{k,n}$  and  $z_{k,n}$  is then of the order  $\frac{\ln n}{n|1 - z_{k,n}|^2}$ . We state our result on the asymptotic expansion of  $z_{k,n}$ .

**Theorem 4.1.** *There exist polynomials  $Q_j(x, y)$  of degree  $j$  in the variable  $y$  and of degree  $\leq 2j - 2$  in the variable  $x$  such that for  $0 < \beta < 1$ ,  $n^\beta < k \leq n/2$ , and  $r \in \mathbb{N}$  we have*

$$z_{k,n} = \alpha_{k,n} \left( 1 + \sum_{j=1}^{r-1} \frac{Q_j(\alpha_{k,n}, \ln[\sqrt{2\pi n}(\alpha_{k,n} - 1)])}{n^j(1 - \alpha_{k,n})^{2j-1}} \right) + O \left[ \left( \frac{\ln n}{n} \right)^r \left( \frac{n}{k} \right)^{r-\frac{1}{2}} \right],$$

where the constant in the error term only depends on the choice of  $\beta$  and  $r$ . The polynomials  $Q_j$  can be computed explicitly. For example, we have

$$Q_1(x, y) = -\frac{1}{2}y; \quad Q_2(x, y) = -\frac{1}{8}y^2 + \frac{1}{2}xy - \frac{1}{12}(x^2 + 10x + 1).$$

For  $r = 1$  (i.e. without correction terms) this result was first proved in [2, (A.47)].

Faster rates of convergence can be achieved by using a better approximation of  $F_n$  in the definition of  $A_n$  that is provided by Theorem 2.2, namely

$$A_n(z) = \tilde{\phi}(z) + \frac{1}{n} \ln(\sqrt{2\pi n}(1 - z)).$$

The faster rate of approximation comes at the price that the corresponding approximate solutions  $\alpha_{k,n}$  now lie on  $n$ -dependent curves  $D_n$  rather than on the Szegő curve. We leave the corresponding expansion of the zeros in terms of such approximate solutions  $\alpha_{k,n}$  for a later publication. Note that both [2, (A.48)] and [4] also work with such better approximations.

**4.2. Zeros near the critical point.** In order to formulate the result we first need to recall the definition of the complementary error function

$$\operatorname{erfc}(z) = 1 - \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt \quad \text{for } z \in \mathbb{C},$$

where the path of integration of the latter integral is subject to the restriction  $\arg(t) \rightarrow \alpha$  with  $|\alpha| < \frac{\pi}{4}$  as  $t \rightarrow \infty$  along the path. It is well known (see e.g. [7]) that all the zeros of this function lie in the second and third quadrant of the complex plane (i.e. in the regions  $\frac{\pi}{2} < \arg(z) < \pi$  and  $-\pi < \arg(z) < -\frac{\pi}{2}$ ) and that in the second quadrant there are countably many zeros of the complementary error function. We denote these zeros by  $w_k, k \in \mathbb{N}$ , and we can order them by modulus  $|w_k| < |w_{k+1}|$ . Our result on the solutions  $z_{k,n}$  of (4.2) reads as follows.

**Theorem 4.2.** *There exist polynomials  $q_j$  of degree  $j$  such that for  $0 < \beta < 1$ ,  $1 < k < n^\beta$ , and  $r \in \mathbb{N}$  we have*

$$z_{k,n} = 1 + \sum_{j=1}^{r-1} \frac{q_j(\sqrt{2}w_k)}{n^{j/2}} + O\left(\left(\frac{k}{n}\right)^{r/2}\right),$$

where the constant in the error term only depends on the choice of  $r$  and  $\beta$ . Moreover, the polynomials  $q_j$  may be computed explicitly, e.g.

$$q_1(x) = x; \quad q_2(x) = \frac{x^2 - 1}{3}; \quad q_3(x) = \frac{x^3 - 7x}{36}.$$

Such a result was proved for  $r = 2$  and  $\beta < 1/3$  in [2, (A.34)] sharpening and extending previous results of [9], [4]. Since  $|w_k| \sim \sqrt{2\pi k}$  for  $k \rightarrow \infty$  it follows that the  $j$ -th term in the expansion above is of order  $(\frac{k}{n})^{j/2}$ . In particular, we have that  $|z_{k,n} - 1|$  is of order  $\sqrt{\frac{k}{n}}$ . Consequently, the expansion of  $z_{k,n}$  in terms of the zeros of the complementary error function holds for all zeros in shrinking circles of size  $n^{(\beta-1)/2}$ . Thus, for any  $\varepsilon > 0$  and  $n^{-(1/2)+\varepsilon} < |z_{k,n} - 1| < n^{-\varepsilon}$  subsections 4.1 and 4.2 provide different expansions for the solutions of (4.2). A short calculation shows that Theorem 4.2 yields better approximations in shrinking circles of size  $O(n^{-1/3})$ , otherwise the approximation with  $\alpha_{k,n}$  on the Szegő curve (see Theorem 4.1) is more advantageous.

We finish by explaining how the complementary error function enters the picture in the proof of Theorem 4.2. Introducing the auxiliary function

$$v(\zeta) := e^{\zeta^2} \operatorname{erfc}(\zeta),$$

one verifies that  $v(\zeta) = 2e^{\zeta^2} - v(-\zeta)$  holds for  $\zeta \in \mathbb{R}$  and hence by the identity principle on all of the complex plane. Since  $v$  and the complementary error function have the same set of roots we obtain  $2e^{w_k^2}v(-w_k)^{-1} = 1$ . Elementary estimates on the arguments show that the correct value of the logarithm is given by

$$(4.4) \quad w_k^2 - \ln(v(-w_k)/2) + 2\pi i k = 0, \quad \text{for all } k \in \mathbb{N}.$$

Next we state the relation between  $v$  and the function  $h$  defined in (3.1). For all  $\zeta$  with positive real part the following relation holds

$$(4.5) \quad v(\zeta) = 2h(i\zeta).$$

To see this one verifies that  $g(\zeta) := 2h(i\zeta)e^{-\zeta^2}$  has the same derivative and the same limiting behavior for  $\zeta \rightarrow \infty$  as the complementary error function. Recall that the function  $h$  was used to define the parametrix  $P_n$  (3.2). Following the procedure described at the beginning of the present section and keeping the result of Theorem 3.3 in mind we approximate  $G_n$  by

$$A_n(z) := \tilde{\phi}(z) - \frac{1}{n} \ln P_n(z).$$

Setting  $\alpha_{k,n} := \lambda(w_k/\sqrt{n})$ , using that  $\phi$  and  $\tilde{\phi}$  agree on the upper half plane, and that the real part of  $-w_k$  is positive, we obtain from (2.3), (4.5), and (4.4)

$$A_n(\alpha_{k,n}) = \phi(\lambda(w_k/\sqrt{n})) - \frac{1}{n} \ln h(-iw_k) = \frac{1}{n} [w_k^2 - \ln(v(-w_k)/2)] = -\frac{2\pi i k}{n}$$

satisfying (4.3) as desired.

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FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT BOCHUM, UNIVERSITÄTSSTR. 150, D-44801 BOCHUM, GERMANY

*E-mail address*: thomas.kriecherbauer@rub.de

DEPARTMENT OF MATHEMATICS, KATHOLIEKE UNIVERSITEIT LEUVEN, CELESTIJNENLAAN 200 B, 3001 LEUVEN, BELGIUM

*E-mail address*: arno.kuijlaars@wis.kuleuven.be

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ARIZONA, TUCSON, AZ 85721, USA

*E-mail address*: mcl@math.arizona.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, 530 CHURCH ST., ANN ARBOR, MI 48109, USA

*E-mail address*: millerpd@umich.edu